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On a New Series of Integrable Nonlinear Evolution Equations

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Recent results of our research are surveyed in this report. The derivative nonlinear Schrödinger equation for the circular polarized Alfvén wave admits the spiky soliton solutions for the plane wave boundary condition. The nonlinear equation for complex amplitude associated with the carrier wave is shown to be a generalized nonlinear Schrödinger equation, having the ordinary cubic nonlinear term and the derivative of cubic nonlinear term. A generalized scheme of the inverse scattering transformation has confirmed that superposition of the A-K-N-S scheme and the K-N scheme for the component equations valids for the generalized nonlinear Schrödinger equation.

Then, two types of new integrable nonlinear evolution equation have been derived from our scheme of the inverse scattering transformation. One is the type of nonlinear Schrödinger equation, while the other is the type of Korteweg-de Vries equation. Brief discussions are presented for physical phenomena, which could be accounted by the second type of the new integrable nonlinear evolution equation.

Lastly, the stationary solitary wave solutions have been constructed for the integrable nonlinear evolution equation of the second type. These solutions have peculiar structure that they are singular and discreet. It is a new challenge to construct singular potentials by the inverse scattering transformation.

§1. Introduction

Plasma under the external magnetic fields are unique media that could sustain many kinds of oscillations, of which amplitude attain easily such a large level that nonlinear effects prevail in competition with dispersive effects of the waves. Basing on the reductive perturbation theory, Taniuti and his collaborators[1)] have derived the Korteweg-de Vries equation for the ion acoustic wave, the cubic nonlinear Schrödinger equation for the electron Langmuir wave and the modified Korteweg-de Vries equation for the Alfvèn wave. Yet, these do not exhaust all of the possible varieties of nonlinear evolution equations describing the nonlinear wave propagation in plasmas. In fact, the derivative nonlinear Schrödinger equation has been derived for the Alfvèn wave propagating parallel to the magnetic field[2)].

It was the great success of Ablowitz, Kaup, Newell and Segur[3)] that they could find a unified scheme of the inverse scattering transformation for the above mentioned canonical type of nonlinear evolution equations. Since the derivative nonlinear Schrödinger equation is unable to be casted into the A-K-N-S scheme, we have carried out detailed analysis of the stationary solution of this new type of nonlinear evolution equation[4),5)]. Then, Kaup and Newell[6)] have presented a new scheme of the inverse scattering transformation for the derivative nonlinear Schrödinger equation.

According to our analysis[5], however, the solutions we have obtained are the stationary solutions of a generalized nonlinear Schrödinger equation, of which nonlinear terms are sum of the cubic nonlinear term and the derivative nonlinear term. Through a generalization of the inverse scattering transformation, we have shown explicitly that the inverse scattering transformation for the generalized nonlinear Schrödinger equation is nothing but a linear superposition of the A-K-N-S scheme and K-N scheme[7].

Furthermore, within the scheme of our generalization of the inverse scattering transformation, we have discovered two types of integrable nonlinear evolution equations, both of which have saturative nonlinear terms[8].

In the present report, we summarize recent development of our studies, and will discuss the novel feature of soliton solutions of these new integrable nonlinear evolution equations.

§2. Spiky Solitary Waves of Alfvén Wave

The circularly polarized Alfvén wave propagating along the magnetic field obeys the derivative nonlinear Schrödinger equation[2]

$$i \frac{\partial}{\partial t} q \pm \mu \frac{\partial^2}{\partial \xi^2} q + i \frac{1}{4} \frac{\partial}{\partial \xi} \{|q|^2 q\} = 0 \quad , \qquad 1)$$

where q is the complex magnetic field

$$q = B_y \mp iB_z \quad , \qquad 2)$$

and ξ is the moving coordinate with the Alfvén velocity in x-direction.

We seek stationary solutions of eq.1) describing the nonlinear self-modulation of large amplitude plane wave[5]. Setting q as

$$q(\xi,t) = \sqrt{8}\psi(\xi,t)\exp[i\chi(\xi,\tau)] \quad , \qquad 3.a)$$

$$\chi(\xi,t) = (k\xi - \Omega t) + \theta(y) \quad , \qquad 3.b)$$

$$\psi(\xi,t) = \psi(y) \quad , \qquad 3.c)$$

with

$$y = \xi - \lambda t \quad , \qquad 3.d)$$

we obtain

$$\psi^2(y) \equiv \Phi(y) = \Phi_0 + 8\kappa\gamma^2\beta^{-1}\{\kappa m + \cosh[2\gamma|\mu|^{-1/2}(y-y_0)]\}^{-1} , \quad 4.a)$$

$$\theta(y) = \theta(y_0) + 3\kappa\frac{|\mu|^{1/2}}{\mu}\arctan\{\sqrt{\frac{1-\kappa m}{1+\kappa m}}\tanh[\gamma|\mu|^{-1/2}(y-y_0)]\}$$
$$+\kappa\delta\frac{|\mu|^{1/2}}{\mu}\arctan\{\sqrt{\frac{1-\kappa \ell}{1+\kappa \ell}}\tanh[\gamma|\mu|^{-1/2}(y-y_0)]\} , \qquad 4.b)$$

where

$$\kappa = \pm 1, \qquad 5.a)$$

$$\delta = \text{sign of } (3\Phi_0 - \lambda - 2\mu k) , \qquad 5.b)$$

$$\alpha = 2\{2\Phi_0 - \lambda - 2(1+\mu)k\} \quad , \qquad 5.c)$$

$$\beta = 4[(\Phi_0 + k)[\lambda + (1+2\mu)k - 2\Phi_0]]^{1/2} \quad , \qquad 5.d)$$

$$\gamma^2 = \frac{1}{4}(\lambda - \lambda_1)(\lambda_2 - \lambda) \quad , \qquad 5.e)$$

$$\ell = \alpha/\beta + 8\gamma^2/(\beta\Phi_0) \quad , \qquad 5.f)$$

$$m = \alpha/\beta \quad .$$

The allowed range of propagation velocity λ is determined to be

$$\lambda_1 < \lambda < \lambda_2 \qquad 6)$$

with the definition of

$$\lambda_1 = 2(2\Phi_0 - \mu k) - 2\sqrt{\Phi_0(\Phi_0 + k)} \qquad 7.a)$$

$$\lambda_2 = 2(2\Phi_0 - \mu k) + 2\sqrt{\Phi_0(\Phi_0 + k)} \qquad 7.b)$$

Referring to the left polarized Alfvén wave, we illustrate characteristic feature of the solitary wave solutions given by eqs.4.a) and 4.b). For arbitrary chosen parameters of $\kappa=+1$, $\sqrt{\Phi_0}=0.5/\sqrt{8}$, $k=0.01$ and $\mu=0.5$, $\lambda=2(2\Phi_0-\mu k)$, Fig.1 represents the bright hyperbolic solitary wave, resulting from the self-modulation of

Fig.1

the large amplitude plane wave, due to a strong coupling of nonlinear modulation of the amplitude and the phase. When the propagation velocity λ takes the limiting velocities λ_1 or λ_2, eqs.(7.a), (7.b), the hyperbolic solitary wave solution is reduced to the algebraic solitary wave solution,

$$\psi^2(y)\equiv\Phi(y)=\Phi_0+\frac{4\rho}{4+\rho^2|\mu|^{-1}(y-y_0)^2} \qquad 8.a)$$

$$\theta(y)=\theta(y_0)+\frac{|\mu|^{1/2}}{\mu}\ \varepsilon\arctan\left(\tfrac{1}{2}\rho\nu|\mu|^{-1/2}(y-y_0)\right)$$

$$+\frac{|\mu|^{1/2}}{\mu}\ 3\arctan\left(\tfrac{1}{2}\rho|\mu|^{-1/2}(y-y_0)\right) \qquad 8.b)$$

where

$$\rho=4(\Phi_0+k)+\varepsilon 4\sqrt{\Phi_0(\Phi_0+k)} \qquad 9.a)$$

$$\nu=\sqrt{\Phi_0}/|2\sqrt{\Phi_0+k}+\varepsilon\sqrt{\Phi_0}| \qquad 9.b)$$

$$\varepsilon=\begin{cases}+1 & \text{for } \lambda=\lambda_2\\ -1 & \text{for } \lambda=\lambda_1\end{cases} \qquad 9.c)$$

Fig. 2 illustrates the fast alegebraic solitary wave moving with the velocity λ_2 for the same values of ψ_0, k and μ as in Fig.1.

Fig.2

In order to show physical differences between the derivative nonlinear Schrödinger equation for the Alfvén wave and the cubic nonlinear Schrödinger equation, we substitute

$$q(\xi,t)=Q(\eta,t)\exp[i(k\eta-\mu k^2t)], \qquad 10.a)$$

with

$$\eta=\xi-2\mu kt \qquad , \qquad 10.b)$$

into eq.(1). Then, we get an equation for the complex amplitude $Q(\eta,t)$ as

$$i\frac{\partial}{\partial t}Q+\mu\frac{\partial^2}{\partial\eta^2}Q-\frac{k}{4}|Q|^2Q+\frac{i}{4}\frac{\partial}{\partial\eta}\{|Q|^2Q\}=0, \tag{11}$$

where the nonlinear terms are composed of the usual cubic nonlinear term and the derivative nonlinear term. The spiky solitary waves presented as eqs.4.a), 4.b) and 8.a), 8.b) are the stationary state solution of eq.11).

§3. A Generalization of Inverse Scattering Method

In spite of the success of Ablowitz, Kaup, Newell and Segur[3)] to present a unified scheme of the inverse scattering transformation for a certain class of nonlinear evolution equations, their scheme appears to be not general enough to cover large varieties of the integrable nonlinear evolution equation. Indeed, Kaup and Newell[6)] have presented another scheme for the derivative nonlinear Schrödinger equation and the massive Thirring equation. Inspired by the solitary wave solutions of the generalized nonlinear Schrödinger equation, eq.(11), we have undertaken to seek a generalization of the inverse scattering transformation to cover a wider class of nonlinear evolution equations[7)].

We consider the eigenvalue problem.

$$\frac{\partial}{\partial x}v_1+F(\lambda)v_1=G(\lambda)q(x,t)v_2 \qquad \text{12.a)}$$

$$\frac{\partial}{\partial x}v_2 - F(\lambda)v_2 = G(\lambda)r(x,t)v_1 \qquad 12.b)$$

where $F(\lambda)$ and $G(\lambda)$ are functions of the eigenvalue λ. The A-K-N-S scheme is a special case of our generalization, $F(\lambda)=i\lambda$ and $G(\lambda)=1$. The time dependence of the eigenfunctions is chosen to be

$$\frac{\partial}{\partial t}v_1 = A(\lambda,q,r)v_1 + B(\lambda,q,r)v_2, \qquad 13.a)$$

$$\frac{\partial}{\partial t}v_2 = C(\lambda,q,r)v_1 - A(\lambda,q,r)v_2. \qquad 13.b)$$

Noting that $(v_{ix})_t=(v_{it})_x$, $i=1,2$, and assuming that the eigenvalues λ are time invariant we readily find that $A(\lambda,q,r)$, $B(\lambda,q,r)$ and $C(\lambda,q,r)$ satisfy

$$A_x + G(rB - qC) = 0 \qquad 14.a)$$

$$Gq_t - B_x - 2FB - 2GqA = 0 \qquad 14.b)$$

$$Gr_t - C_x + 2FC + 2GrA = 0 \qquad 14.c)$$

The proper choices of A, B, C, F and G yield various integrable nonlinear evolution equations.

Referring to the A-K-N-S scheme and the K-N scheme, we choose

$$F(\lambda) = i\alpha\lambda^2 - \sqrt{2\beta}\,\lambda \;, \qquad 15.a)$$

$$G(\lambda) = \alpha\lambda + i\sqrt{\beta/2} \quad , \qquad 15.b)$$

Namely we consider the superposition of these two scheme of the inverse scattering transformation. Here, α and β are positive constants. Then, with the choice of

$$A(\lambda,q,r)=-2i\alpha^2\lambda^4+4\alpha\sqrt{2\beta}\lambda^3+(4i\beta-i\alpha^2rq)\lambda^2$$
$$+\sqrt{2\beta}\alpha rq\lambda+i(\beta/2)rq \quad , \tag{16.a}$$
$$B(\lambda,q,r)=2\alpha^2q\lambda^3+3i\sqrt{2\beta}\alpha q\lambda^2+(-2\beta q+i\alpha q_x+\alpha^2rq^2)\lambda$$
$$+(-\sqrt{\beta/2}q_x+i\alpha\sqrt{\beta/2}rq^2), \tag{16.b}$$
$$C(\lambda,q,r)=2\alpha^2r\lambda^3+3i\sqrt{2\beta}\alpha r\lambda^2+(-2\beta r-i\alpha r_x+\alpha^2r^2q)\lambda$$
$$+(\sqrt{\beta/2}r_x+i\alpha\sqrt{\beta/2}r^2q), \tag{16.c}$$

we obtain from eqs.(14.b) and (14.c) the set of nonlinear evolution equations

$$iq_t+q_{xx}-i\alpha(rq^2)_x+\beta rq^2=0 \quad , \tag{17.a}$$
$$ir_t-r_{xx}-i\alpha(r^2q)_x-\beta r^2q=0. \tag{17.b}$$

If we take $r=\pm q^*$, the set of equations (17.a) and (17.b) is reduced to the generalized nonlinear Schrödinger equation

$$iq_t+q_{xx}\mp i\alpha(|q|^2q)_x\pm\beta|q|^2q=o. \tag{18}$$

The above analysis is the first illustration of the fact that the linear superposition of two different scheme of the inverse scattering transformation works out to solve the nonlinear evolution equation with mixed type of nonlinear terms.

§4. New Integrable Nonlinear Evolution Equations

Having been encouraged by the success of our generalization of the inverse scattering transformation, we have undertaken to explore new types of integrable nonlinear evolution equations[8].

We assign for $F(\lambda)$ and $G(\lambda)$ of eqs.(12.a) and (12.b) the following expressions,

$$F(\lambda) = i\lambda \quad , \tag{19.a}$$

$$G(\lambda) = \lambda \quad . \tag{19.b}$$

Then, choosing

$$A = -\frac{2i}{\sqrt{1-rq}}\lambda^2 \ , \tag{20.a}$$

$$B = 2\frac{q}{\sqrt{1-rq}}\lambda^2 + i\left(\frac{q}{\sqrt{1-rq}}\right)_x\lambda \ , \tag{20.b}$$

$$C = 2\frac{r}{\sqrt{1-rq}}\lambda^2 - i\left(\frac{r}{\sqrt{1-rq}}\right)_x\lambda \ , \tag{20.c}$$

We obtain from eqs.(14.b) and (14.c)

$$q_t - i\left(\frac{q}{\sqrt{1-rq}}\right)_{xx} = 0 \tag{21.a}$$

$$r_t + i\left(\frac{r}{\sqrt{1-rq}}\right)_{xx} = 0 \tag{21.b}$$

If we take $r=\pm q^*$, eqs.(21.a) and (21.b) are reduced to

$$iq_t + \frac{\partial^2}{\partial x^2}\left\{\frac{q}{\sqrt{1\mp|q|^2}}\right\} = 0 \ . \tag{22}$$

Another choices of A, B and C,

$$A = -\frac{4i}{\sqrt{1-rq}}\lambda^3 + \frac{rq_x - qr_x}{(1-rq)^{3/2}}\lambda^2 \ , \tag{23.a}$$

$$B = \frac{4q}{\sqrt{1-rq}}\lambda^3 + \frac{2iq_x}{(1-rq)^{3/2}}\lambda^2 - \left(\frac{q_x}{(1-rq)^{3/2}}\right)_x\lambda \quad , \tag{23.b}$$

$$C = \frac{4r}{\sqrt{1-rq}}\lambda^3 - \frac{2ir_x}{(1-rq)^{3/2}}\lambda^2 - \left(\frac{r_x}{(1-rq)^{3/2}}\right)_x\lambda \quad , \tag{23.c}$$

lead to the following set of equations

$$q_t + \left(\frac{\partial^2}{\partial x^2}\frac{1}{(1-rq)^{3/2}}\frac{\partial q}{\partial x}\right) = 0 \ , \tag{24.a}$$

$$r_t + \left(\frac{\partial^2}{\partial x^2}\frac{1}{(1-rq)^{3/2}}\frac{\partial r}{\partial x}\right) = 0 \ . \tag{24.b}$$

If we take r=-q, the set of eqs.(24.a) and (24.b) is reduced to

$$\frac{\partial}{\partial t}q + \frac{\partial^2}{\partial x^2}\left(\frac{1}{(1+q^2)^{3/2}}\frac{\partial q}{\partial x}\right) = 0. \tag{25}$$

We notice also, for r=-1 and q=u-1, eqs.(24.a) and (24.b) are reduced to

$$\frac{\partial}{\partial t}u = 2\frac{\partial^3}{\partial x^3}u^{-1/2}, \tag{26}$$

which is the equation discovered by Harry Dym[9].

Shimizu and Wadati[10] have carried out detailed analysis of eq.(22) with the lower sign under the boundary condition

$$q(x,t) \to 0 \quad \text{as } |x| \to \infty \ . \tag{27}$$

For the bound state eigenvalue λ_1 given as

$$\lambda_1 = \xi + i\eta \quad , \qquad \eta>0 \quad , \tag{28}$$

an expression for one-soliton solution is obtained;

$$q(x,t)=-\frac{2\eta}{\sqrt{\xi^2+\eta^2}}\;\frac{\cosh[2\eta y+2\eta\varepsilon_+(y)-i\alpha]}{\cosh^2[2\eta y+2\eta\varepsilon_+(y)]-2\frac{\eta^2}{\xi^2+\eta^2}}$$

$$\times\ \exp[-2i\{\xi x+2(\xi^2-\eta^2)t+\xi\varepsilon_+(y)\}], \tag{29.a}$$

where $\varepsilon_+(y)$ is given implicityly as

$$y=-\varepsilon_+(y)+\frac{1}{4\eta}\log\left[-\frac{2\eta}{\xi^2+\eta^2}\frac{1}{\varepsilon_+(y)}-1\right]. \tag{29.b}$$

The quantity y is a moving coordinate defined by

$$y = x + 4\xi t. \tag{30}$$

As for the illustration, the function $\varepsilon_+(y)$ is shown in Fig. 3 for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$.

Fig.3

Since $\varepsilon_+(y)$ is determined numerically from eq.(29.b), we can evaluate q(x,t) numerically. Figure 4 illustrates the one soliton envelope $|q(x,t)|$ for $\eta=1/2$ with various values

Fig.4

of ξ/η. For the small amplitude limit of $\eta<<\xi$, eqs.(29.a) and (29.b) are reduced to

$$q(x,t)=\frac{2\eta}{\xi}\mathrm{sech}[2\eta(x+4\xi t)]\exp[-2i\xi x-4i(\xi^2-\eta^2)t], \quad 31)$$

which is the one soliton solution of the cubic nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2\xi^2|q|^2q = 0. \quad 32)$$

In Fig.4, the limit of a bursting soliton given as $|\xi|\to\eta+0$ has been shown, as well.

§5. Some Physical Problems Related to Eq.(25)

Although the new integrable nonlinear evolution equations bear novel feature for the mathematical interest, we have tried to identify physical problems for which the new integrable evolution equations are relevant. We find the eq.(25) could be the key equation for physical problems in which the curvature of surface or any deformation is crucial.

One of such problems is the nonlinear transverse oscillation of elastic beams under tension[11]. We can write down the equations of motion of the small

Fig.5

element AB illustrated in Fig.5 as

$$\rho A \frac{\partial^2}{\partial t^2} y = \frac{\partial}{\partial x} S, \quad 33.a)$$

$$0 = \frac{\partial}{\partial x} M + P \frac{\partial}{\partial x} y + S \; ; \qquad 33.b)$$

where ρ is density of material, A stands for area of cross section. S is the stress resultant parallel to the axis of y, and P is the end-thrust parallel to the axis of x. As far as we are considering a uniform elastic beam, P is assumed to be constant.

For the bending moment M, we have the relation

$$M = \frac{EI}{R} = EI \frac{1}{\{1+(\partial y/\partial x)^2\}^{3/2}} \frac{\partial^2 y}{\partial x^2} \, , \qquad 34)$$

where E is Young's modulus, R represents the radius of curvature of bending beam, and I is the moment of inertia of the cross section of beam. When the beam subjects to tension $P=-\mathcal{J}$, combining eqs.(33.a), (33.b) and (34), we obtain the following nonlinear partial differential equation

$$\frac{\partial^2}{\partial t^2} y - \frac{\mathcal{J}}{\rho A} \frac{\partial^2}{\partial x^2} y + \frac{EI}{\rho A} \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial^2 y/\partial x^2}{[1+(\partial y/\partial x)^2]^{3/2}} \right\} = 0 . \qquad 35)$$

Since we are dealing with the uniform elastic beam, we do not have any dynamical nonlinear effect, but we have taken fully account of the geometrical nonlinear effect.

Introducing the dimensionless variables X, Y and T as

$$x = A^{1/2} X \qquad 36.a)$$

$$y = A^{1/2} Y \qquad 36.b)$$

$$t = (A^{1/2}/\lambda) \, T, \qquad 36.c)$$

where $\lambda = (\mathcal{J}/\rho A)^{1/2}$ is the linear wave velocity, and defining the stretched coordinates

$$\xi = X + T \tag{37.a}$$

$$\tau = \varepsilon T \tag{37.b}$$

with the dimensionless parameter ε given as

$$\varepsilon = \frac{EI}{2\mathcal{T}A}, \tag{38}$$

we can reduce eq.(35) to, up to the first order of ε,

$$\frac{\partial}{\partial\tau}\frac{\partial}{\partial\xi} Y + \frac{\partial^2}{\partial\xi^2}\left\{\frac{\partial^2 Y/\partial\xi^2}{[1+(\partial Y/\partial\xi)^2]^{3/2}}\right\} = 0. \tag{39}$$

Defining

$$q(\xi,\tau) = \frac{\partial Y}{\partial\xi}, \tag{40}$$

we can see immediately that eq.(39) is nothing but the equation of the second types of our new integrable nonlinear evolution equations, that is

$$\frac{\partial}{\partial\tau}q + \frac{\partial^2}{\partial\xi^2}\left\{\frac{\partial q/\partial\xi}{(1+q^2)^{3/2}}\right\} = 0. \tag{41}$$

Another example could be found in the surface phenomena in which the surface tension plays the key role. For instance[12] the shape of the surface of a fluid in a gravitational field and bounded on one side by a vertical plane wall is determined by

$$\frac{2}{a^2} Z - \frac{Z''}{(1+Z'^2)^{3/2}} = \text{const}, \tag{42}$$

where

$$a = \sqrt{(2\alpha/\rho g)} \tag{43}$$

is called the capillary constant. α is the surface tension coefficient, and g is the gravitational acceleration.

Defining the function q(x) by

$$q = \frac{\partial z}{\partial x}, \qquad 44)$$

and differentiating eq.(42) twice with respect to x, we obtain

$$\frac{2}{a^2}\frac{\partial}{\partial x} q - \frac{\partial^2}{\partial x^2}\{\frac{q'}{(1+q^2)^{3/2}}\} = 0. \qquad 45)$$

Therefore, eq.(43) could be regarded as the stationary solution of eq.(41), which depends on the single variable

$$\eta = \xi - \frac{2}{a^2}\tau. \qquad 46)$$

Of course, in order to apply the inverse scattering method to integrate eqs.(41) or (45), we have to carefully consider the boundary conditions of the problems.

§6. Analysis of Stationary Solutions of Eq.(41)

Here, let us examine stationary solutions of eq.(41) assuming that q depends on a single parameter

$$\eta = \xi \pm v\tau, \quad v > 0. \qquad 47)$$

Substituting the transformation (47) into eq.(41), we can carry out the first integration to get

$$vq \pm \frac{d}{d\eta}\{(1+q^2)^{-3/2}\frac{d}{d\eta}q\} = 0 \qquad 48)$$

for the boundary conditions of q=0 and $dq/d\eta=0$ at $|\eta|\to\infty$.
Setting

$$q = \frac{dY}{d\eta}, \qquad 49)$$

we can again integrate eq.(48) to yield

$$vY \pm \{1 + (\frac{dY}{d\eta})^2\}^{-3/2} \frac{d^2}{d\eta^2} Y = 0. \quad 50)$$

Multiplying $dY/d\eta$ on the both side of eq.(50), we can get

$$\frac{v}{2} Y^2 \mp \{1 + (\frac{dY}{d\eta})^2\}^{-1/2} = C. \quad 51)$$

For the lower sign of eq.(51), we can obtain a localized solution by imposing the boundary conditions at the infinity as

$$\frac{dY}{d\eta} = 0, \quad Y = 0 \quad \text{at} \quad \eta \to \pm\infty. \quad 52)$$

This boundary condition specifies the value of C as

$$C = 1. \quad 53)$$

We assign the following boundary conditions at a point η_0,

$$\frac{dY}{d\eta} = -\cot\theta, \quad Y = Y_0 \quad \text{at} \quad \eta \to \eta_0 + 0, \quad 54.a)$$

$$\frac{dY}{d\eta} = +\cot\theta, \quad Y = Y_0 \quad \text{at} \quad \eta \to \eta_0 - 0, \quad 54.b)$$

where θ is an arbitrary constant. For a given value of θ, Y_0 is determined to be

$$Y_0 = \pm(\frac{2}{v})^{1/2} (1 - \sin\theta)^{1/2}. \quad 55)$$

For the positive sign of eq.(55), carrying out the integration of $dY/d\eta$ derived from eq.(51), we obtain

$$\pm\sqrt{v}(\eta-\eta_0) = -\text{sech}^{-1}|\frac{\sqrt{v}}{2}Y| + 2(1 - \frac{v}{4} Y^2)^{1/2}$$
$$+\text{sech}^{-1}|\frac{\sqrt{v}}{2}Y_0| - 2(1 - \frac{v}{4}Y_0{}^2)^{1/2}. \quad 56)$$

In Fig.6, we illustrate the solution $Y(\eta)$

Fig.6

for $\theta=0$, $\theta=\pi/4$ and $\theta=\pi/3$. The solitary wave solution of eq. (41), $q(\eta)$, is calculated from eq.(49) with eq.(51). Corresponding to the

Fig.7

three cases of Fig.6, we illustrate in Fig.7 the corresponding solutions. These are singular and discrete solitary waves. Although such singular solutions are not acceptable as physical solutions for the nonlinear transverse oscillation of the elastic beam, the present analysis poses the challenging problem whether we can construct such solutions by the inverse scattering transformation.

§7. Concluding Discussions

In the present report, we have surveyed recent results of our investigation on the nonlinear evolution equations. Generalization of the inverse scattering transformation has demonstrated that the linear superposition of two schemes of the inverse scattering transformation works to solve the generalized nonlinear evolution equation with the superposed nonlinear terms, each of which corresponds the component scheme of the inverse scattering transformation. We need to explore to what extent such superposition works out to solve the nonlinear evolution equation.

In this connection, it may be worth to mention the work of Hirota[13). He had shown that the generalized nonlinear equation

$$iq_t+\rho q_{xx}+i\sigma q_{xxx}+\delta|q|^2q+i3\alpha|q|^2q_x=0 \qquad 57)$$

has N-soliton solutions, provided that the condition

$$\alpha\rho = \sigma\delta \qquad 58)$$

holds. Eq.(57) is a superposition of the nonlinear Schrödinger equation and the complex modified Korteweg-de Vries equation. For the special case of $\rho=0$ and $\delta=0$, eq.(57) is known to have multiple envelope soliton solutions[14).. On the other hand, Chen, Lee and Liu[15) have shown that eq.(57) is integrable for the special case of $\sigma=0$ and $\delta=0$, which violates the condition eq.(58).

Studies of the new integrable nonlinear evolution equation have revealed the new problem of the inverse scattering transformation to construct singular potentials. If the singular solitary waves and the discrete solitary waves demonstrated in the section 6 could obtain the citizen-ship of the soliton empire, the extension of soliton science will be far more reaching than that was expected when Zabusky and Kruskal[16) had coined the concept of soliton for the solitary wave of the Korteweg-de Vries equation.

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Captions of Figures

Fig.1 The bright hyperbolic solitary wave of the left polarized Alfvén wave for a value of $\lambda=2(2\Phi_0-\mu k)$ with an arbitrary chosen phase constant $\theta(y_0)=\pi/6$. The thin line represents the magnetic field component B_y and the dotted line stands for $|B|$, respectively.

Fig.2 The fast algebraic solitary wave of the left polarized Alfvén wave with a propagation velocity λ_2.

Fig.3 The curve of $\varepsilon_+(u)$ for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$.

Fig.4 The envelope of one-soliton solution for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$, $\sqrt{2}$ and 1.

Fig.5 Transverse displacement of elastic beam under the end-thrust.

Fig.6 Localized stationary solutions of eq.(39) for $\theta=0$, $\theta=\pi/4$ and $\theta=\pi/3$.

Fig.7 Singular solitary wave solutions of eq.(41) derived from the solution Y with $\theta=0$, $\theta=\pi/4$ and $\theta=\pi/3$, respectively.

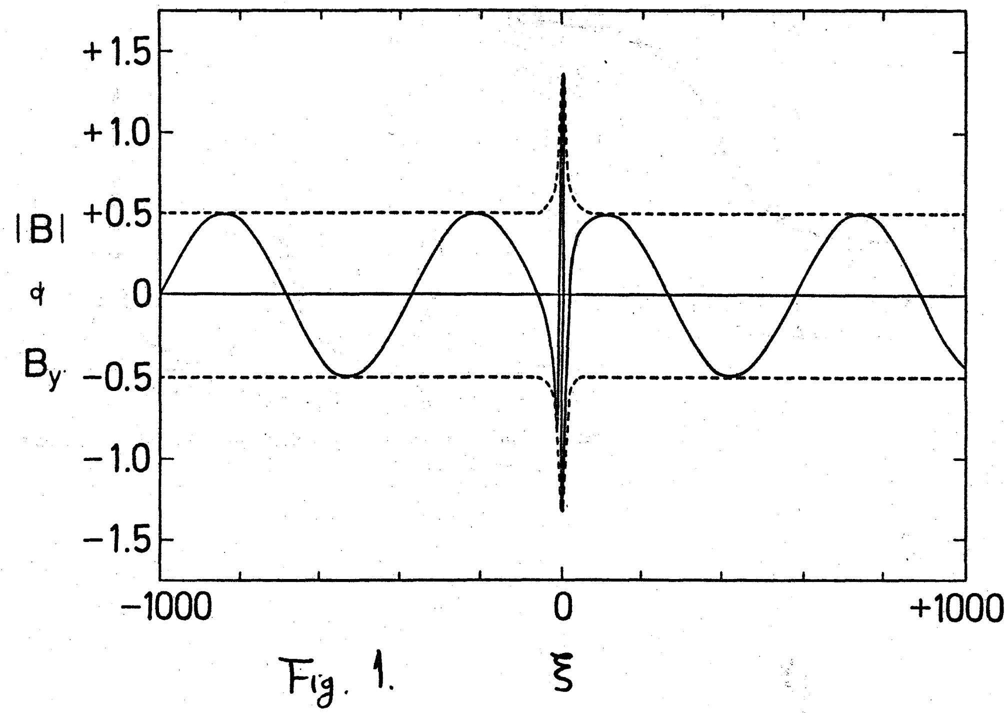

+1.5
+1.0
+0.5
0
−0.5
−1.0
−1.5
|B|
&
B_y
−1000
0
+1000
ξ

Fig. 1.

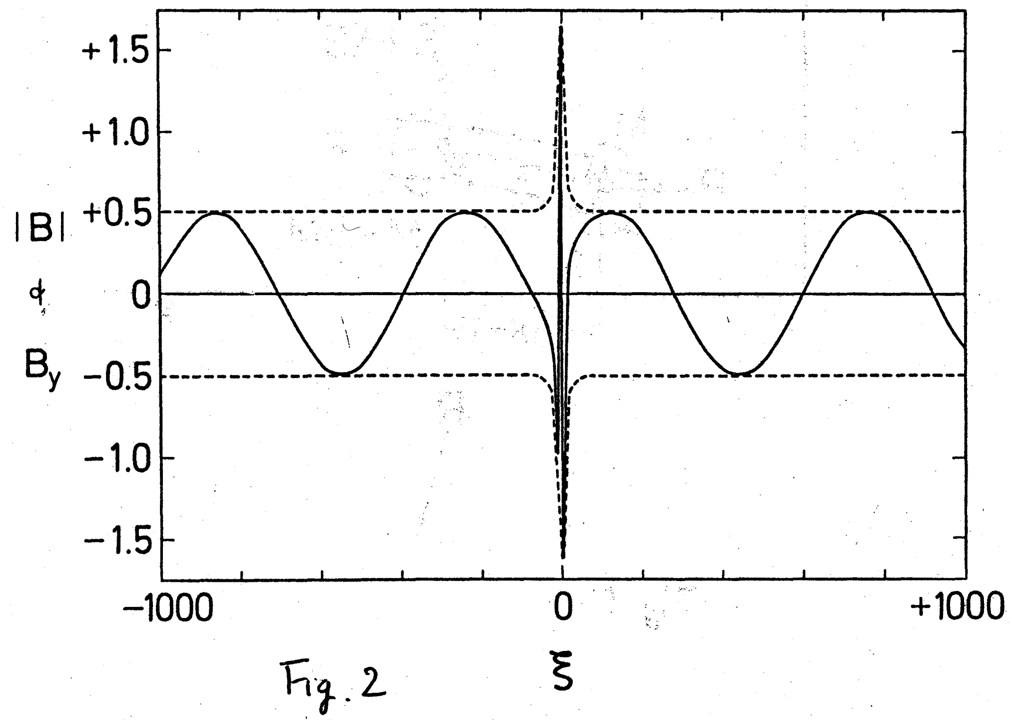

+1.5
+1.0
+0.5
0
−0.5
−1.0
−1.5
|B|
&
B_y
−1000
0
+1000
ξ

Fig. 2

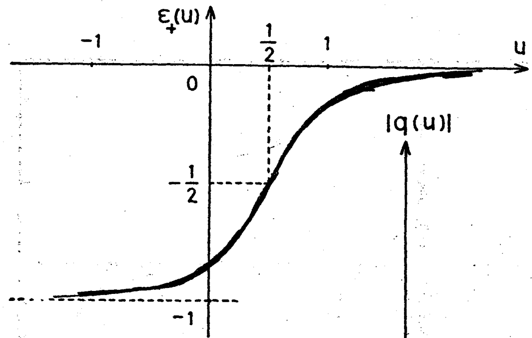


Fig. 3 The curve of $\varepsilon_+(u)$ for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$.

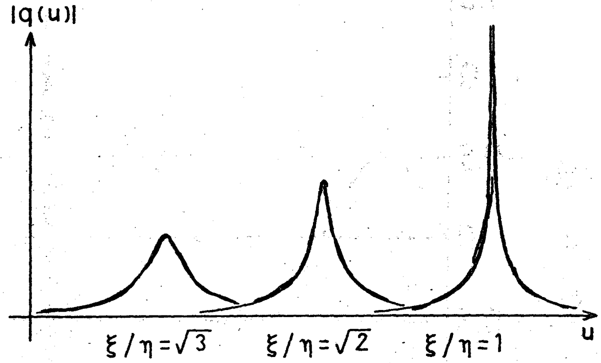


Fig. 4 The envelope of one-soliton solution for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$, $\sqrt{2}$ and 1.

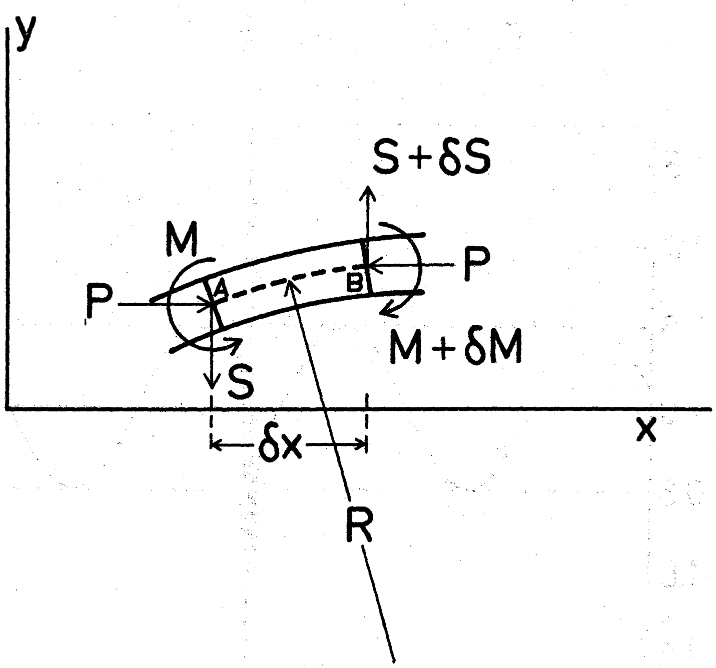
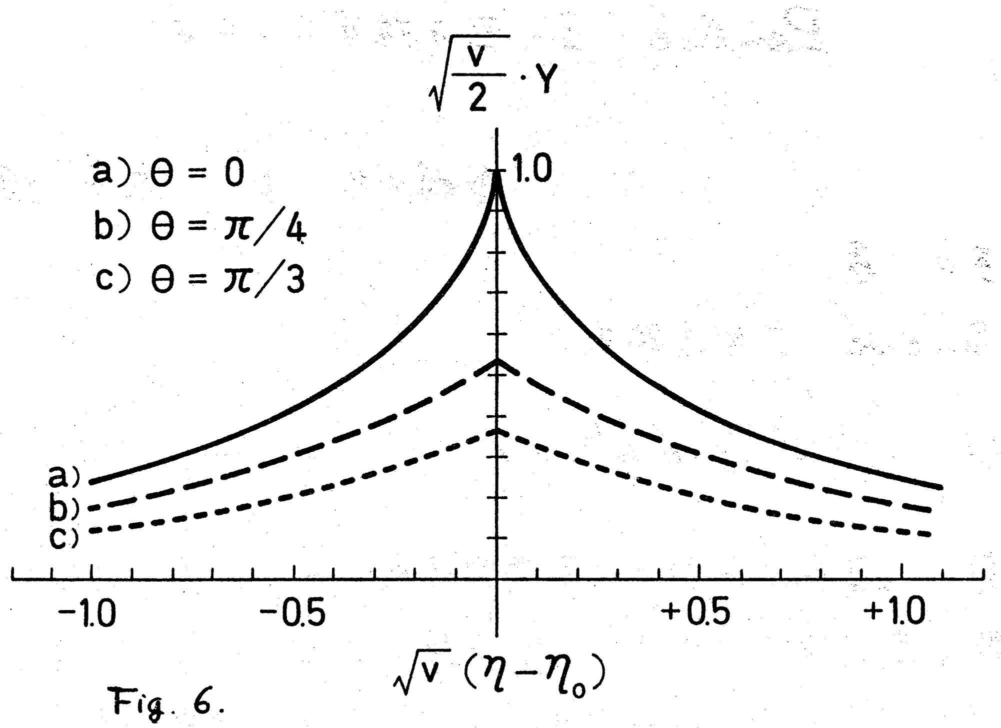
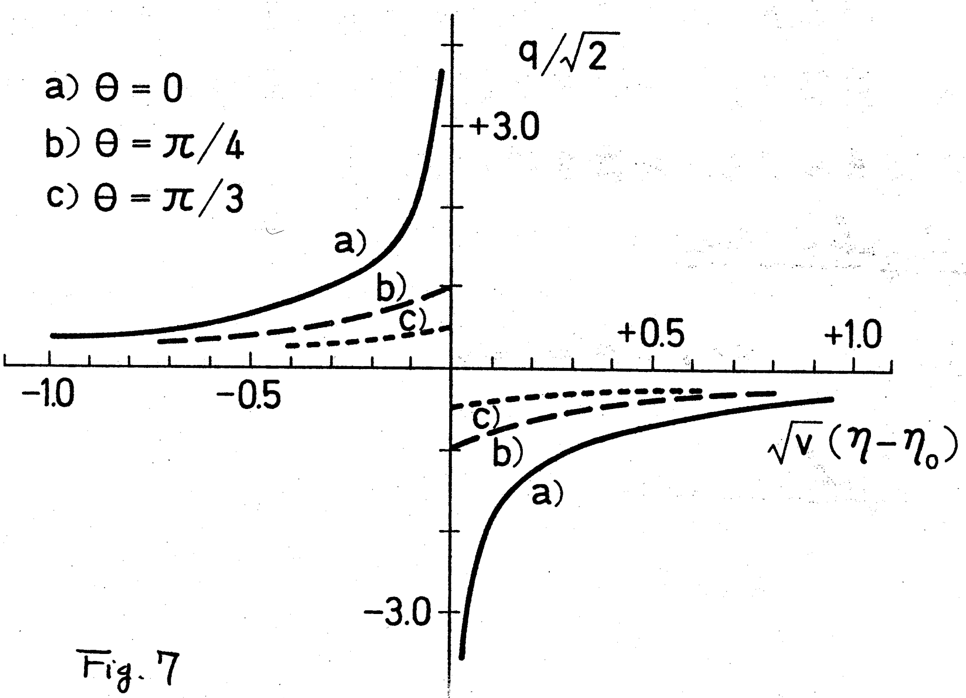


Fig. 5



$\sqrt{\frac{v}{2}} \cdot Y$
a) $\Theta = 0$
b) $\Theta = \pi/4$
c) $\Theta = \pi/3$
1.0
a)
b)
c)
−1.0
−0.5
+0.5
+1.0
$\sqrt{v}\,(\eta - \eta_o)$

Fig. 6.



$q/\sqrt{2}$
a) $\Theta = 0$
b) $\Theta = \pi/4$
c) $\Theta = \pi/3$
+3.0
a)
b)
c)
+0.5
+1.0
−1.0
−0.5
c)
b)
a)
$\sqrt{v}\,(\eta - \eta_o)$
−3.0

Fig. 7